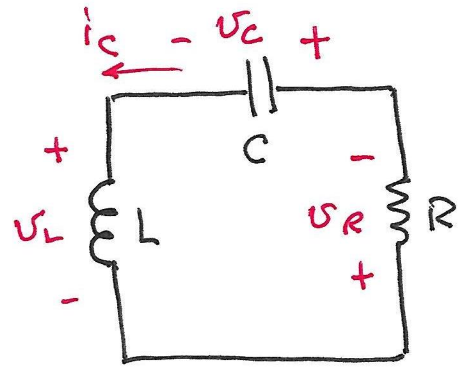
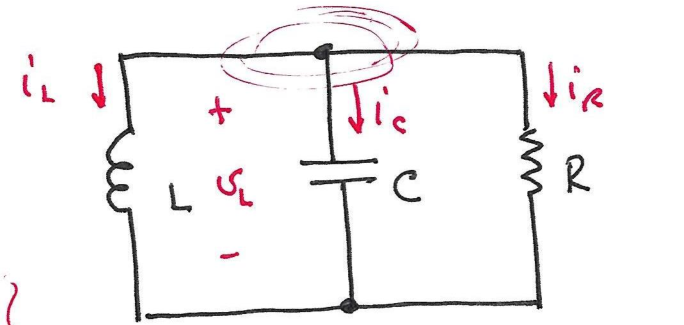
RLC Circuits





$$i_C = C\dot{v}_C$$

$$v_L = L\frac{di_C}{dt} = LC\ddot{v}_C$$

$$v_R = R i_C = RC\dot{v}_C$$

KVL:

$$v_L + v_R + v_C = 0$$

$$LC\ddot{v}_C + RC\dot{v}_C + v_C = 0$$

$$\Rightarrow \ddot{v}_C + \frac{R}{L}\dot{v}_C + \frac{1}{LC}v_C = 0$$

$$v_L = L\frac{di_L}{dt}$$

$$i_C = C\dot{v}_L = CL\frac{d^2 i_L}{dt^2}$$

$$i_R = \frac{v_L}{R} = \frac{L}{R}\frac{di_L}{dt}$$

KCL:

$$i_C + i_R + i_L = 0$$

$$CL\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = 0$$

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = 0$$

Both are described by a 2^{nd} order ODE of the form:

$$a\ddot{x} + b\dot{x} + cx = 0$$

Guess that $x = Ke^{\alpha t}$

$$a\alpha^2 Ke^{\alpha t} + b\alpha Ke^{\alpha t} + cKe^{\alpha t} = 0$$

$$a\alpha^2 + b\alpha + c = 0$$

Characteristic Equation

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = LC$, $b = RC$, $c = 1$

or

$a = 1$, $b = \frac{R}{L}$, $c = \frac{1}{LC}$

$a = LC$, $b = \frac{L}{R}$, $c = 1$

or

$a = 1$, $b = \frac{1}{RC}$, $c = \frac{1}{LC}$

Case 1: If $b^2 > 4ac$, then $\sqrt{b^2-4ac}$ is a positive real number.

Both roots are real, and distinct.

$$\alpha_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad \alpha_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\therefore \quad x = K_1 e^{\alpha_1 t} + K_2 e^{\alpha_2 t}$$

Case 2: If $b^2-4ac = 0$ or $b^2 = 4ac$

$$\alpha = \alpha_1 = \alpha_2 = \frac{-b}{2a}$$

$$x = (K_1 + K_2 t)\, e^{\alpha t}$$

Case 3 : $\quad b = 0$

Then

$$\alpha = \frac{\pm\sqrt{-4ac}}{2a}$$

$$\alpha = \pm\sqrt{-d^2} = \pm \underset{\substack{\uparrow \\ \sqrt{-1}}}{j}\, d$$

$$x = K_1 e^{jdt} + K_2 e^{-jdt}$$

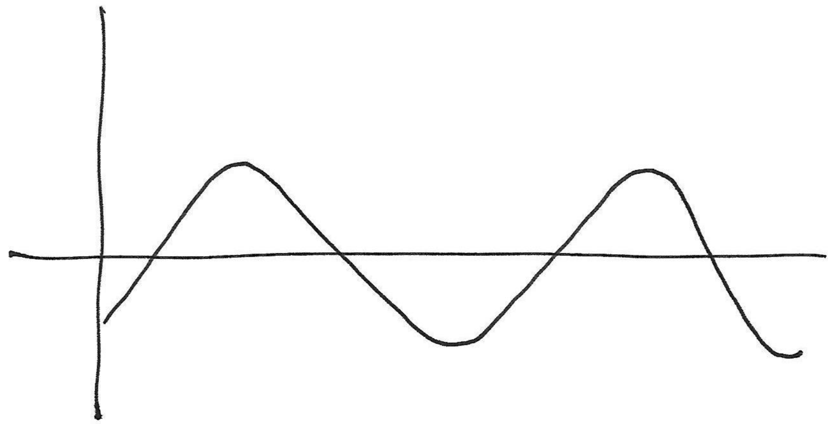
Euler's Identity :

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\therefore \quad x = K_1(\cos dt + j\sin dt) + K_2(\cos dt - j\sin dt)$$

$$= \underbrace{(K_1 + K_2)}_{\text{real}} \cos dt + \underbrace{j(K_1 - K_2)}_{\text{real}} \sin dt$$

$$= K_3 \cos dt + K_4 \sin dt$$



Case 4:

$$\alpha_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \alpha_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

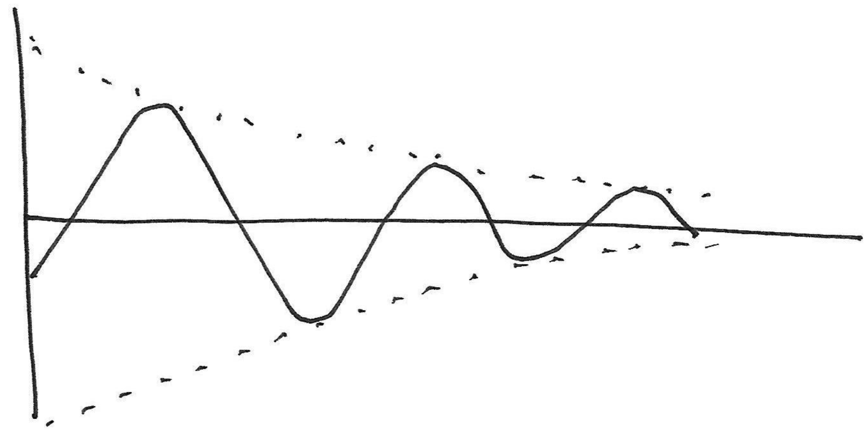
$b^2 < 4ac$, so that the α's are complex

$$\alpha_{1,2} = e \pm jd$$

$$x = K_1 e^{(e+jd)t} + K_2 e^{(e-jd)t}$$

$$= e^{et}\left[K_1 e^{jdt} + K_2 e^{-jdt}\right]$$

$$= e^{et}\left[K_3 \cos dt + K_4 \sin dt\right]$$



Example

$$\alpha^2 + 3\alpha + 2 = 0 \qquad \text{characteristic equation}$$

$$(\alpha + 1)(\alpha + 2) = 0$$

$$\boxed{x = K_1 e^{-1t} + K_2 e^{-2t}}$$

The usual way to determine the K's is to use initial conditions for x and $\dot{x}$

Assume $x(0) = 1$

$\dot{x}(0) = 0$

$$\boxed{\dot{x} = -1K_1 e^{-1t} - 2K_2 e^{-2t}}$$

$$x(0) = K_1 + K_2 = 1$$

$$\dot{x}(0) = -K_1 - 2K_2 = 0$$

$$K_2 - 2K_2 = 1 + 0$$

$$K_2 = -1$$

$$\therefore \ K_1 = 2$$

$$\Rightarrow \quad x = 2e^{-t} - e^{-2t} \quad , \quad t \geq 0$$

Example 2

$$\ddot{x} + 4 = 0 \qquad x(0) = 10$$

$$\dot{x}(0) = 4$$

$$(x + j2)(x - j2) = 0$$

$$x = K_1 \cos 2t + K_2 \sin 2t$$

$$\dot{x} = -2K_1 \sin 2t + 2K_2 \cos 2t$$

$$x(0) = K_1 = 10$$

$$\dot{x}(0) = 2K_2 = 4$$

$$\therefore \quad x(t) = 10 \cos 2t + 2 \sin 2t \quad , \quad t \geq 0$$

Example 3

$$\ddot{x} + 4\dot{x} + 13x = 0 \qquad x(0) = 0$$

$$\dot{x}(0) = 1$$

$$\alpha^2 + 4\alpha + 13 = 0$$

$$(\alpha + 2) + 9 = 0$$

$$(\alpha + 2) + 3^2 = 0$$

$$\Rightarrow \quad \alpha = -2 \pm j3$$

$$x = K_1 e^{(-2+j3)t} + K_2 e^{(-2-j3)t}$$

$$= e^{-2t}\left(K_1 e^{j3t} + K_2 e^{-j3t}\right)$$

$$x = e^{-2t}\left[(K_1 + K_2)\cos 3t + j(K_1 - K_2)\sin 3t\right]$$

$$\dot{x} = -2e^{-2t}\left[(K_1 + K_2)\cos 3t + j(K_1 - K_2)\sin 3t\right]$$
$$+ e^{-2t}\left[-3(K_1 + K_2)\sin 3t + j3(K_1 - K_2)\cos 3t\right]$$

$$x(0) = K_1 + K_2 = 0$$

$$\dot{x}(0) = -2(K_1 + K_2) + j3(K_1 - K_2) = 1$$

Solve for K_1 and K_2